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# A vertex operator approach for form factors of Belavin's $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model\*

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## Abstract

Belavin's  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model is considered on the basis of bosonization of vertex operators in the  $A_{n-1}^{(1)}$  model and vertex-face transformation. Free-field representations of nonlocal tail operators are constructed for off-diagonal matrix elements with respect to the ground state sectors. As a result, integral formulae for form factors of any local operators in the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model can, in principle, be obtained.

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## 1. Introduction

This paper is a continuation of [1], in which we derived the integral formulae for correlation functions of Belavin's  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [2, 3] on the basis of vertex operator approach [4]. Belavin's  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model is an  $n$ -state generalization of Baxter's eight-vertex model [5], which has  $(\mathbb{Z}/2\mathbb{Z})$ -symmetries. As for the eight-vertex model, the integral formulae for correlation functions were derived by Lashkevich and Pugai [6], and those for form factors were derived by Lashkevich [7].

It was found in [6] that the correlation functions of the eight-vertex model can be obtained by using the free-field realization of the vertex operators in the eight-vertex SOS model [8], with insertion of the nonlocal operator  $\Lambda$ , called 'the tail operator'. The most essential part of [6] was the construction of free-field representations of  $\Lambda$ 's. Furthermore, those of the off-diagonal (with respect to the ground state sector) elements of  $\Lambda$ 's were constructed in [7], in order to obtain the form factor formulae of the eight-vertex model.

There are some researches which generalize the study of [6]. The vertex operator approach for higher spin generalization of the eight-vertex model was presented in [9]. For higher rank generalization, the integral formulae for correlation functions of Belavin's  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model were presented in our previous paper [1]. The expression of the spontaneous polarization

\* Dedicated to the memory of my parents.

of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [10] was also reproduced in [1], on the basis of the vertex operator approach. To the best of our knowledge, there has not been a developed research of [7] related to the form factor problem. The aim of the present paper is to give a higher rank generalization of the bosonization scheme in the eight-vertex model.

The present paper is organized as follows. In section 2 we review the basic definitions of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [2], the corresponding dual face model  $A_{n-1}^{(1)}$ -model [11] and the vertex-face correspondence. Some detailed definitions of the models concerned are listed in appendix A. In section 3 we introduce the type I and type II vertex operators of both  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and the  $A_{n-1}^{(1)}$ -model, and also introduce the tail operators. Furthermore, we derive the commutation relations to which those operators should satisfy. In order to obtain integral formulae for form factors of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model we construct the free-field representations of off-diagonal elements of the tail operators, by using those of the type I [12] and type II [13, 14] vertex operators in the  $A_{n-1}^{(1)}$ -model in section 4. Useful operator product expansion (OPE) formulae and commutation relations for basic bosons are given in appendix B. In section 5 we give some concluding remarks. Among these remarks, a brief proof of the commutation relations of the type I and the type II vertex operators in the  $A_{n-1}^{(1)}$ -model is given in appendix C.

## 2. Basic definitions

The present section aims to formulate the problem, thereby fixing the notation.

### 2.1. Theta functions

The Jacobi theta function with two pseudo-periods 1 and  $\tau$  ( $\text{Im } \tau > 0$ ) is defined as follows:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v; \tau) := \sum_{m \in \mathbb{Z}} \exp\{\pi \sqrt{-1}(m+a)[(m+a)\tau + 2(v+b)]\}, \quad (2.1)$$

for  $a, b \in \mathbb{R}$ . Let  $n \in \mathbb{Z}_{\geq 2}$  and  $r \in \mathbb{R}$  such that  $r > n - 1$ , and also fix the parameter  $x$  such that  $0 < x < 1$ . We will use the abbreviations,

$$[v] = x^{\frac{v^2}{r}-v} \Theta_{x^{2r}}(x^{2v}), \quad [v]' = x^{\frac{v^2}{r-1}-v} \Theta_{x^{2r-2}}(x^{2v}), \quad (2.2)$$

where

$$\Theta_q(z) = (z; q)_\infty (qz^{-1}; q)_\infty (q; q)_\infty = \sum_{m \in \mathbb{Z}} q^{m(m-1)/2} (-z)^m,$$

$$(z; q_1, \dots, q_m)_\infty = \prod_{i_1, \dots, i_m \geq 0} (1 - zq_1^{i_1} \dots q_m^{i_m}).$$

Note that

$$\vartheta \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \left( \frac{v}{r}, \frac{\pi \sqrt{-1}}{\epsilon r} \right) = \sqrt{\frac{\epsilon r}{\pi}} \exp\left(-\frac{\epsilon r}{4}\right) [v],$$

where  $x = e^{-\epsilon}$  ( $\epsilon > 0$ ).

For later conveniences we also introduce the following symbols:

$$r_j(v) = z^{\frac{r-1}{r} \frac{n-j}{n}} \frac{g_j(z^{-1})}{g_j(z)}, \quad g_j(z) = \frac{\{x^{2n+2r-j-1}z\}\{x^{j+1}z\}}{\{x^{2n-j+1}z\}\{x^{2r+j-1}z\}}, \quad (2.3)$$

$$r_j^*(v) = z^{\frac{r}{r-1} \frac{n-j}{n}} \frac{g_j^*(z^{-1})}{g_j^*(z)}, \quad g_j^*(z) = \frac{\{x^{2n+2r-j-1}z\}'\{x^{j-1}z\}'}{\{x^{2n-j-1}z\}'\{x^{2r+j-1}z\}'}, \quad (2.4)$$

$$\chi_j(v) = z^{-\frac{j(a-j)}{n}} \frac{\rho_j(z^{-1})}{\rho_j(z)}, \quad \rho_j(z) = \frac{(-x^{2j+1}z; x^2, x^{2n})_\infty (-x^{2n-2j+1}z; x^2, x^{2n})_\infty}{(-xz; x^2, x^{2n})_\infty (-x^{2n+1}z; x^2, x^{2n})_\infty}, \quad (2.5)$$

where  $z = x^{2v}$ ,  $1 \leq j \leq n$  and

$$\{z\} = (z; x^{2r}, x^{2n})_\infty, \quad \{z\}' = (z; x^{2r-2}, x^{2n})_\infty. \quad (2.6)$$

In particular, we denote  $\chi(v) = \chi_1(v)$ . These factors will appear in the commutation relations among the type I and type II vertex operators.

The integral kernel for the type I and the type II vertex operators will be given as the products of the following elliptic functions:

$$f(v, w) = \frac{[v + \frac{1}{2} - w]}{[v - \frac{1}{2}]}, \quad h(v) = \frac{[v - 1]}{[v + 1]}, \quad (2.7)$$

$$f^*(v, w) = \frac{[v - \frac{1}{2} + w]'}{[v + \frac{1}{2}]'}, \quad h^*(v) = \frac{[v + 1]'}{[v - 1]'} \quad (2.8)$$

In section 4, we use the following identities:

$$\sum_{v=0}^{n-1} \prod_{\substack{j=0 \\ j \neq v}}^{n-1} \frac{f(v_{j+1} - v_j, 1 - p_v + p_j)}{[p_v - p_j]} = 0 \quad (2.9)$$

and

$$\sum_{v=0}^{n-1} \prod_{\substack{j=0 \\ j \neq v}}^{n-1} \frac{f^*(v_j - v_{j+1}, 1 - p_j + p_v)}{[p_v - p_j]'} = 0, \quad (2.10)$$

where  $v_n = v + \frac{n}{2}$  and  $\sum_{j=0}^{n-1} p_j = 0$ . The former one (2.9) was derived in [12] by applying Liouville's second theorem to the following elliptic function:

$$F(w) = \prod_{j=0}^{n-1} \frac{[v_{j+1} - v_j - \frac{1}{2} + w - p_j]}{[v_{j+1} - v_j - \frac{1}{2}][w - p_j]}.$$

The latter one (2.10) can be similarly proved.

### 2.2. $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and its dual face model

Let  $V = \mathbb{C}^n$  and  $\{\varepsilon_\mu\}_{0 \leq \mu \leq n-1}$  be the standard orthonormal basis with the inner product  $\langle \varepsilon_\mu, \varepsilon_\nu \rangle = \delta_{\mu\nu}$ . Belavin's  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model [2] is a vertex model on a two-dimensional square lattice  $\mathcal{L}$  such that the state variables take the values of  $(\mathbb{Z}/n\mathbb{Z})$ -spin. The model is  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric in a sense that the  $R$ -matrix satisfies the following conditions:

$$\begin{aligned} \text{(i)} \quad & R(v)_{jl}^{ik} = 0, \text{ unless } i + k = j + l, \text{ mod } n, \\ \text{(ii)} \quad & R(v)_{j+pl+p}^{i+pk+p} = R(v)_{jl}^{ik}, \quad \forall i, j, k, l, p \in \mathbb{Z}/n\mathbb{Z}. \end{aligned} \quad (2.11)$$

The  $R$ -matrix satisfies the Yang–Baxter equation (YBE)

$$R_{12}(v_1 - v_2)R_{13}(v_1 - v_3)R_{23}(v_2 - v_3) = R_{23}(v_2 - v_3)R_{13}(v_1 - v_3)R_{12}(v_1 - v_2), \quad (2.12)$$

where  $R_{ij}(v)$  denotes the matrix on  $V^{\otimes 3}$ , which acts as  $R(v)$  on the  $i$ th and  $j$ th components and as identity on the other one. As for the elliptic parametrization of  $R$ -matrix, see appendix A.

The dual face model of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model is called the  $A_{n-1}^{(1)}$ -model. This is a face model on a two-dimensional square lattice  $\mathcal{L}^*$ , the dual lattice of  $\mathcal{L}$ , such that the state variables take the values of the dual space of Cartan subalgebra  $\mathfrak{h}^*$  of  $A_{n-1}^{(1)}$ :

$$\mathfrak{h}^* = \bigoplus_{\mu=0}^{n-1} \mathbb{C}\omega_\mu, \tag{2.13}$$

where

$$\omega_\mu := \sum_{\nu=0}^{\mu-1} \bar{\varepsilon}_\nu, \quad \bar{\varepsilon}_\mu = \varepsilon_\mu - \frac{1}{n} \sum_{\mu=0}^{n-1} \varepsilon_\mu.$$

The weight lattice  $P$  and the root lattice  $Q$  of  $A_{n-1}^{(1)}$  are usually defined, see appendix A.

An ordered pair  $(a, b) \in \mathfrak{h}^{*2}$  is called *admissible* if  $b = a + \bar{\varepsilon}_\mu$ , for a certain  $\mu$  ( $0 \leq \mu \leq n - 1$ ). For  $(a, b, c, d) \in \mathfrak{h}^{*4}$ , let  $W \left[ \begin{smallmatrix} c & d \\ b & a \end{smallmatrix} \middle| v \right]$  be the Boltzmann weight of the  $A_{n-1}^{(1)}$  model for the state configuration  $\left[ \begin{smallmatrix} c & d \\ b & a \end{smallmatrix} \right]$  round a face. Here the four states  $a, b, c$  and  $d$  are ordered clockwise from the SE corner. In this model  $W \left[ \begin{smallmatrix} c & d \\ b & a \end{smallmatrix} \middle| v \right] = 0$  unless the four pairs  $(a, b), (a, d), (b, c)$  and  $(d, c)$  are admissible. Non-zero Boltzmann weights are given by (A.6)–(A.8), see appendix A.

Among those, the weight (A.7) is different from the corresponding one used in our previous paper [1] by a minus sign. Accordingly, in the present paper we will use different definitions of the intertwining vectors (2.15) and the type I vertex operators (4.5), (4.6) from the corresponding objects of [1] by extra factors of the form  $(-1)^A$ 's. This difference simply results from a gauge transformation.

The Boltzmann weights solve the Yang–Baxter equation for the face model [11]:

$$\begin{aligned} \sum_g W \left[ \begin{smallmatrix} d & e \\ c & g \end{smallmatrix} \middle| v_1 \right] W \left[ \begin{smallmatrix} c & g \\ b & a \end{smallmatrix} \middle| v_2 \right] W \left[ \begin{smallmatrix} e & f \\ g & a \end{smallmatrix} \middle| v_1 - v_2 \right] \\ = \sum_g W \left[ \begin{smallmatrix} g & f \\ b & a \end{smallmatrix} \middle| v_1 \right] W \left[ \begin{smallmatrix} d & e \\ g & f \end{smallmatrix} \middle| v_2 \right] W \left[ \begin{smallmatrix} d & g \\ c & b \end{smallmatrix} \middle| v_1 - v_2 \right]. \end{aligned} \tag{2.14}$$

### 2.3. Vertex-face correspondence

Let

$$\begin{aligned} t(v)_{a-\bar{\varepsilon}_\mu}^a &= t(v; \epsilon, r)_{a-\bar{\varepsilon}_\mu}^a = \sum_{\nu=0}^{n-1} \varepsilon_\nu t^\nu(v)_{a-\bar{\varepsilon}_\mu}^a, \\ t^\nu(v)_{a-\bar{\varepsilon}_\mu}^a &= \prod_{j=\mu+1}^{n-1} (-1)^{a_{\mu j}} \vartheta \left[ \begin{smallmatrix} \frac{n}{2} \\ \frac{1}{2} + \frac{\nu}{n} \end{smallmatrix} \right] \left( \frac{v}{nr} + \frac{\bar{a}_\mu}{r}; \frac{\pi\sqrt{-1}}{n\epsilon r} \right) \end{aligned} \tag{2.15}$$

be the intertwining vectors. (See appendix A, concerning the definition of  $\bar{a}_\mu$ .) Then  $t(v)_{a-\bar{\varepsilon}_\mu}^a$ 's relate the  $R$ -matrix of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model in the principal regime and Boltzmann weights  $W$  of the  $A_{n-1}^{(1)}$ -model in the so-called regime III (cf figure 1),

$$R(v_1 - v_2) t(v_1)_a^d \otimes t(v_2)_d^c = \sum_b t(v_1)_b^c \otimes t(v_2)_a^b W \left[ \begin{smallmatrix} c & d \\ b & a \end{smallmatrix} \middle| v_1 - v_2 \right]. \tag{2.16}$$

Note that the present intertwining vectors are different from the ones used in [11], which relate the  $R$ -matrix of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model in the disordered phase and Boltzmann weights  $W$  of the  $A_{n-1}^{(1)}$ -model in the regime III.

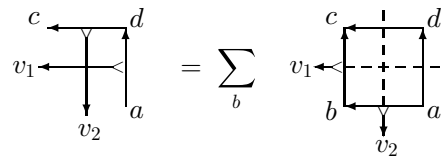


Figure 1. A picture representation of the vertex-face correspondence.

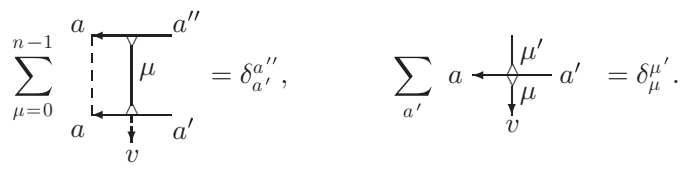


Figure 2. A picture representation of the dual intertwining vectors.

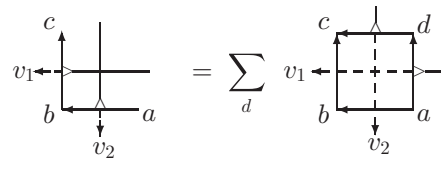


Figure 3. Vertex-face correspondence by dual intertwining vectors.

Let us introduce the dual intertwining vectors (see figure 2) satisfying

$$\sum_{\mu=0}^{n-1} t_{\mu}^*(v)_a^{a'} t^{\mu}(v)_{a''}^a = \delta_{a''}^{a'}, \quad \sum_{\nu=0}^{n-1} t^{\nu}(v)_{a-\bar{\epsilon}_{\nu}}^a t_{\mu'}^*(v)_a^{a-\bar{\epsilon}_{\nu}} = \delta_{\mu'}^{\mu}. \tag{2.17}$$

From (2.16) and (2.17), we have (cf figure 3)

$$t^*(v_1)_c^b \otimes t^*(v_2)_b^a R(v_1 - v_2) = \sum_d W \left[ \begin{matrix} c & d \\ b & a \end{matrix} \middle| v_1 - v_2 \right] t^*(v_1)_d^a \otimes t^*(v_2)_c^d. \tag{2.18}$$

For fixed  $r > n - 1$ , let

$$S(v) = -R(v)|_{r \rightarrow r-1}, \quad W' \left[ \begin{matrix} c & d \\ b & a \end{matrix} \middle| v \right] = -W \left[ \begin{matrix} c & d \\ b & a \end{matrix} \middle| v \right] \Big|_{r \rightarrow r-1} \tag{2.19}$$

and

$$t'^*(u)_a^b := t^*(u; \epsilon, r - 1)_a^b. \tag{2.20}$$

Then we have

$$t'^*(v_1)_c^b \otimes t'^*(v_2)_b^a S(v_1 - v_2) = \sum_d W' \left[ \begin{matrix} c & d \\ b & a \end{matrix} \middle| v_1 - v_2 \right] t'^*(v_1)_d^a \otimes t'^*(v_2)_c^d. \tag{2.21}$$

### 3. Vertex operator algebra

#### 3.1. Vertex operators for the $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model

Let  $\mathcal{H}^{(i)}$  be the  $\mathbb{C}$ -vector space spanned by the half-infinite pure tensor vectors of the forms  $\varepsilon_{\mu_1} \otimes \varepsilon_{\mu_2} \otimes \varepsilon_{\mu_3} \otimes \dots$  with  $\mu_j \in \mathbb{Z}/n\mathbb{Z}$ ,  $\mu_j = i + 1 - j \pmod{n}$  for  $j \gg 0$ . (3.1)

Let  $\mathcal{H}^{*(i)}$  be the dual of  $\mathcal{H}^{(i)}$  spanned by the half-infinite pure tensor vectors of the forms  $\dots \otimes \varepsilon_{\mu_{-2}} \otimes \varepsilon_{\mu_{-1}} \otimes \varepsilon_{\mu_0}$  with  $\mu_j \in \mathbb{Z}/n\mathbb{Z}$ ,  $\mu_j = i + 1 - j \pmod{n}$  for  $j \ll 0$ . (3.2)

Then introduce the type I vertex operator by the following half-infinite transfer matrix:

$$\Phi^\mu(v_1 - v_2) = \begin{array}{c} \begin{array}{ccccccc} & & & & & & \dots \\ & & & & & & | \\ & & & & & & \mu \\ & & & & & & | \\ & & & & & & \leftarrow v_1 \\ & & & & & & | \\ & & & & & & \downarrow \\ & & & & & & v_2 \\ & & & & & & | \\ & & & & & & \downarrow \\ & & & & & & v_2 \\ & & & & & & | \\ & & & & & & \downarrow \\ & & & & & & v_2 \\ & & & & & & | \\ & & & & & & \downarrow \\ & & & & & & v_2 \\ & & & & & & \dots \end{array} \end{array} \quad (3.3)$$

Then the operator (3.3) is an intertwiner from  $\mathcal{H}^{(i)}$  to  $\mathcal{H}^{(i+1)}$ . The type I vertex operators satisfy the following commutation relation:

$$\Phi^\mu(v_1)\Phi^{\nu'}(v_2) = \sum_{\mu', \nu''} R(v_1 - v_2)_{\mu\nu'}^{\mu'\nu''} \Phi^{\nu''}(v_2)\Phi^{\mu'}(v_1). \quad (3.4)$$

When we consider an operator related to ‘creation–annihilation’ process, we need another type of vertex operators, the type II vertex operators that satisfy the following commutation relations:

$$\Psi_\nu^*(v_2)\Psi_\mu^*(v_1) = \sum_{\mu', \nu'} \Psi_{\mu'}^*(v_1)\Psi_{\nu'}^*(v_2)S(v_1 - v_2)_{\mu\nu'}^{\mu'\nu'}, \quad (3.5)$$

$$\Phi^\mu(v_1)\Psi_\nu^*(v_2) = \chi(v_1 - v_2)\Psi_\nu^*(v_2)\Phi^\mu(v_1). \quad (3.6)$$

Let

$$\rho^{(i)} = x^{2nH_{\text{CTM}}} : \mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)}, \quad (3.7)$$

where  $H_{\text{CTM}}$  is the CTM Hamiltonian defined in [1]. Then we have the homogeneity relation

$$\Phi^\mu(v)\rho^{(i)} = \rho^{(i+1)}\Phi^\mu(v - n), \quad \Psi_\mu^*(v)\rho^{(i)} = \rho^{(i+1)}\Psi_\mu^*(v - n). \quad (3.8)$$

#### 3.2. Vertex operators for the $A_{n-1}^{(1)}$ -model

For  $k = a + \rho, l = \xi + \rho$  and  $0 \leq i \leq n - 1$ , let  $\mathcal{H}_{l,k}^{(i)}$  be the space of admissible paths  $(a_0, a_1, a_2, \dots)$  such that

$$\begin{aligned} a_0 &= a, & a_j - a_{j+1} &\in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{n-1}\} && \text{for } j = 0, 1, 2, 3, \dots, \\ & & a_j &= \xi + \omega_{i+1-j} && \text{for } j \gg 0. \end{aligned} \quad (3.9)$$

Also, let  $\mathcal{H}_{l,k}^{*(i)}$  be the space of admissible paths  $(\dots, a_{-2}, a_{-1}, a_0)$  such that

$$\begin{aligned} a_0 &= a, & a_j - a_{j+1} &\in \{\bar{\varepsilon}_0, \bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{n-1}\}, && \text{for } j = -1, -2, -3, \dots, \\ & & a_j &= \xi + \omega_{i+1-j} && \text{for } j \ll 0. \end{aligned} \quad (3.10)$$

Introduce the type I vertex operator by the following half-infinite transfer matrix:

$$\Phi(v_1 - v_2)_a^{a+\bar{\varepsilon}_\mu} = v_1 \leftarrow \begin{array}{c} \begin{array}{cccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ v_2 & v_2 & v_2 & \dots \end{array} \\ \begin{array}{c} a+\bar{\varepsilon}_\mu \\ \vdots \\ a \end{array} \end{array} \quad (3.11)$$

Then the operator (3.11) is an intertwiner from  $\mathcal{H}_{l,k}^{(i)}$  to  $\mathcal{H}_{l,k+\bar{\varepsilon}_\mu}^{(i+1)}$ . The type I vertex operators satisfy the following commutation relation:

$$\Phi(v_1)_b^c \Phi(v_2)_a^b = \sum_d W \left[ \begin{array}{c|c} c & d \\ b & a \end{array} \middle| v_1 - v_2 \right] \Phi(v_2)_d^c \Phi(v_1)_a^d. \quad (3.12)$$

The free-field realization of  $\Phi(v_2)_a^b$  was constructed in [12]. See section 4.2.

The type II vertex operators should satisfy the following commutation relations:

$$\Psi^*(v_2)_{\xi_a}^{\xi_c} \Psi^*(v_1)_{\xi_a}^{\xi_d} = \sum_{\xi_b} \Psi^*(v_1)_{\xi_b}^{\xi_c} \Psi^*(v_2)_{\xi_a}^{\xi_b} W' \left[ \begin{array}{c|c} \xi_c & \xi_d \\ \xi_b & \xi_a \end{array} \middle| v_1 - v_2 \right], \quad (3.13)$$

$$\Phi(v_1)_a^{a'} \Psi^*(v_2)_{\xi}^{\xi'} = \chi(v_1 - v_2) \Psi^*(v_2)_{\xi}^{\xi'} \Phi(v_1)_a^{a'}. \quad (3.14)$$

Let

$$\rho_{l,k}^{(i)} = G_a x^{2nH_{l,k}^{(i)}}, \quad (3.15)$$

where

$$G_a = \prod_{0 \leq \mu < \nu \leq n-1} [a_{\mu\nu}].$$

Then we have the homogeneity relation

$$\Phi(v)_a^{a'} \frac{\rho_{a+\rho,l}^{(i)}}{G_a} = \frac{\rho_{a'+\rho,l}^{(i+1)}}{G_{a'}} \Phi(v-n)_a^{a'}, \quad \Psi^*(v)_{\xi}^{\xi'} \rho_{k,\xi'+\rho}^{(i)} = \rho_{k,\xi'+\rho}^{(i+1)} \Psi^*(v-n)_{\xi}^{\xi'}. \quad (3.16)$$

The free-field realization of  $\Psi^*(v)_{\xi}^{\xi'}$  was constructed in [13, 14]. See section 4.3.

### 3.3. Tail operators and commutation relations

In [1] we introduced the intertwining operators between  $\mathcal{H}^{(i)}$  and  $\mathcal{H}_{l,k}^{(i)}$  ( $k = l + \omega_i \pmod{Q}$ ):

$$\begin{aligned} T(u)_{\xi a_0}^{\xi a_0} &= \prod_{j=0}^{\infty} t^{\mu_j} (-u)_{a_{j+1}}^{a_j} : \mathcal{H}^{(i)} \rightarrow \mathcal{H}_{l,k}^{(i)}, \\ T(u)_{\xi a_0} &= \prod_{j=0}^{\infty} t_{\mu_j}^* (-u)_{a_j}^{a_{j+1}} : \mathcal{H}_{l,k}^{(i)} \rightarrow \mathcal{H}^{(i)}, \end{aligned} \quad (3.17)$$

which satisfy

$$\rho^{(i)} = \left( \frac{(x^{2r-2}; x^{2r-2})_{\infty}}{(x^{2r}; x^{2r})_{\infty}} \right)^{(n-1)(n-2)/2} \frac{1}{G'_{\xi}} \sum_{\substack{k \equiv l + \omega_i \\ \pmod{Q}}} T(u)_{a\xi} \rho_{l,k}^{(i)} T(u)^{a\xi}, \quad (3.18)$$

and the intertwining relations



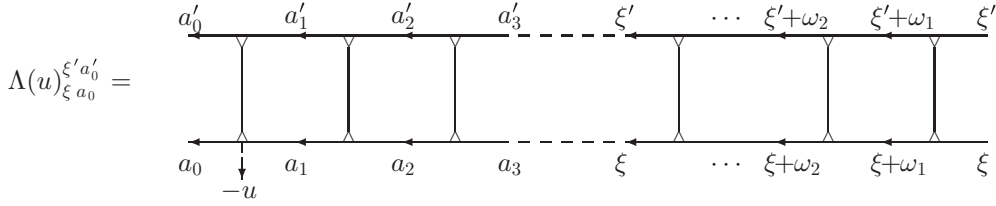


Figure 4. Tail operator  $\Lambda(u)_{\xi a_0}^{\xi' a'_0}$ . The upper (resp. lower) half stands for  $T(u)^{\xi' a'_0}$  (resp.  $T(u)_{\xi a_0}$ ).

$$T(u)^{\xi b} \Phi^\mu(v) = \sum_a t^\mu(v-u)_a^b \Phi(v)_a^b T(u)^{\xi a}, \tag{3.19}$$

$$T(u)_{\xi b} \Phi(v)_a^b = \sum_\mu t_\mu^*(v-u)_b^a \Phi^\mu(v) T(u)_{\xi a}. \tag{3.20}$$

Here,  $k = a_0 + \rho$  and  $l = \xi + \rho$ , and  $0 < \Re(u) < \frac{n}{2} + 1$ .

In order to obtain the form factors of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model, we need the free-field representations of the tail operator which is off-diagonal with respect to the boundary conditions (see figure 4)

$$\Lambda(u)_{\xi a}^{\xi' a'} = T(u)^{\xi' a'} T(u)_{\xi a} : \mathcal{H}_{l, k}^{(i)} \rightarrow \mathcal{H}_{l', k'}^{(i)}, \tag{3.21}$$

where  $k = a + \rho, l = \xi + \rho, k' = a' + \rho$ , and  $l' = \xi' + \rho$ . Let

$$L \left[ \begin{matrix} a'_0 & a'_1 \\ a_0 & a_1 \end{matrix} \middle| u \right] := \sum_{\mu=0}^{n-1} t_\mu^*(-u)_{a_0}^{a_1} t^\mu(-u)_{a'_1}^{a'_0}. \tag{3.22}$$

Then we have

$$\Lambda(u)_{\xi a_0}^{\xi' a'_0} = \prod_{j=0}^{\infty} L \left[ \begin{matrix} a'_j & a'_{j+1} \\ a_j & a_{j+1} \end{matrix} \middle| u \right]. \tag{3.23}$$

Note that

$$L \left[ \begin{matrix} a' & a' - \bar{\varepsilon}_v \\ a & a - \bar{\varepsilon}_\mu \end{matrix} \middle| u \right] = \frac{[u + \bar{a}_\mu - \bar{a}'_v]}{[u]} \prod_{j \neq \mu} \frac{[\bar{a}'_v - \bar{a}_j]}{[a_{\mu j}]}. \tag{3.24}$$

It is obvious from (2.17), we have

$$L \left[ \begin{matrix} a & a' \\ a & a'' \end{matrix} \middle| u \right] = \delta_{a''}^{a'}. \tag{3.25}$$

We therefore have

$$\Lambda(u)_{\xi a}^{\xi' a} = \delta_{\xi}^{\xi'}. \tag{3.26}$$

From (3.19), (3.20) and the definition of the tail operator (3.21) we have

$$\Lambda(u)_{\xi b}^{\xi' c} \Phi(v)_a^b = \sum_d L \left[ \begin{matrix} c & d \\ b & a \end{matrix} \middle| u - v \right] \Phi(v)_d^c \Lambda(u)_{\xi a}^{\xi' d}. \tag{3.27}$$

Consider the algebra

$$\Psi^*(v)_{\xi'}^{\xi} T(u)^{\xi a} = \sum_\mu T(u)^{\xi' a} \Psi_\mu^*(v) t'^\mu(v-u-\Delta u)_{\xi}^{\xi'}, \tag{3.28}$$

$$\Psi_{\mu}^*(v)T(u)_{\xi a} = \sum_{\xi'} T(u)_{\xi' a} \Psi^*(v)_{\xi}^{\xi'} t'_{\mu}(v-u-\Delta u)_{\xi'}^{\xi}. \tag{3.29}$$

From these, we have

$$\Psi^*(v)_{\xi d}^{\xi_c} \Lambda(u)_{\xi_a a'}^{\xi_b} = \sum_{\xi_b} L' \left[ \begin{matrix} \xi_c & \xi_d \\ \xi_b & \xi_a \end{matrix} \middle| u + \Delta u - v \right] \Lambda(u)_{\xi_b a'}^{\xi_c} \Psi^*(v)_{\xi_a}^{\xi_b}, \tag{3.30}$$

where

$$L' \left[ \begin{matrix} \xi_c & \xi_d \\ \xi_b & \xi_a \end{matrix} \middle| u \right] = L \left[ \begin{matrix} \xi_c & \xi_d \\ \xi_b & \xi_a \end{matrix} \middle| u \right] \Big|_{r \rightarrow r-1}. \tag{3.31}$$

We should find a representation of  $\Lambda(u)_{\xi_a}^{\xi' a'}$  and fix the constant  $\Delta u$  that solves (3.27) and (3.30).

### 4. Free-field realization

One of the most standard ways to calculate correlation functions and form factors is the vertex operator approach [4] on the basis of free-field representation. The free-field representations for the type I vertex operators of the  $A_{n-1}^{(1)}$  model were constructed in [12], in terms of oscillators introduced in [15, 16]. Those for the type II vertex operators were constructed in [13, 14], also in terms of oscillators introduced in [15, 16]. It was shown in [17, 18] that the elliptic algebra  $U_{q,p}(\widehat{\mathfrak{sl}}_N)$  provides the Drinfeld realization of the face type elliptic quantum group  $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$  tensored by a Heisenberg algebra. Using these representations we derive the free-field representation of the tail operator in this section.

#### 4.1. Bosons

Let us consider the bosons  $B_m^j$  ( $1 \leq j \leq n-1, m \in \mathbb{Z} \setminus \{0\}$ ) with the commutation relations

$$[B_m^j, B_{m'}^k] = \begin{cases} m \frac{[(n-1)m]_x [(r-1)m]_x}{[nm]_x [rm]_x} \delta_{m+m',0}, & (j = k) \\ -m x^{\text{sgn}(j-k)nm} \frac{[m]_x [(r-1)m]_x}{[nm]_x [rm]_x} \delta_{m+m',0}, & (j \neq k), \end{cases} \tag{4.1}$$

where the symbol  $[a]_x$  stands for  $(x^a - x^{-a})/(x - x^{-1})$ . Define  $B_m^n$  by

$$\sum_{j=1}^n x^{-2jm} B_m^j = 0.$$

Then the commutation relations (4.1) holds for all  $1 \leq j, k \leq n$ . These oscillators were introduced in [15, 16].

For  $\alpha, \beta \in \mathfrak{h}^*$  let us define the zero mode operators  $P_\alpha, Q_\beta$  with the commutation relations

$$[P_\alpha, \sqrt{-1}Q_\beta] = \langle \alpha, \beta \rangle, \quad [P_\alpha, B_m^j] = [Q_\beta, B_m^j] = 0.$$

We will deal with the bosonic Fock spaces  $\mathcal{F}_{l,k}$ , ( $l, k \in \mathfrak{h}^*$ ) generated by  $B_{-m}^j$  ( $m > 0$ ) over the vacuum vectors  $|l, k\rangle$  :

$$\mathcal{F}_{l,k} = \mathbb{C}[\{B_{-1}^j, B_{-2}^j, \dots\}_{1 \leq j \leq n}] |l, k\rangle,$$

where

$$\begin{aligned} B_m^j |l, k\rangle &= 0 (m > 0), \\ P_\alpha |l, k\rangle &= \langle \alpha, \beta_1 k + \beta_2 l \rangle |l, k\rangle, \\ |l, k\rangle &= \exp(\sqrt{-1}(\beta_1 Q_k + \beta_2 Q_l)) |0, 0\rangle, \end{aligned}$$

where  $\beta_1$  and  $\beta_2$  are defined by

$$t^2 - \beta_0 t - 1 = (t - \beta_1)(t - \beta_2), \quad \beta_0 = \frac{1}{\sqrt{r(r-1)}}, \quad \beta_1 < \beta_2. \quad (4.2)$$

#### 4.2. Type I vertex operators

Let us define the basic operators for  $j = 1, \dots, n - 1$

$$U_{-\alpha_j}(v) = \exp(-\beta_1(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j} \log z)) : \exp\left(\sum_{m \neq 0} \frac{1}{m} (B_m^j - B_m^{j+1})(x^j z)^{-m}\right) :, \quad (4.3)$$

$$U_{\omega_j}(v) = \exp(\beta_1(\sqrt{-1}Q_{\omega_j} + P_{\omega_j} \log z)) : \exp\left(-\sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^j x^{(j-2k+1)m} B_m^k z^{-m}\right) :, \quad (4.4)$$

where  $\beta_1 = -\sqrt{\frac{r-1}{r}}$  and  $z = x^{2v}$  as usual. For some useful OPE formulae and commutation relations, see appendix B.

In the following we set

$$\pi_\mu = \sqrt{r(r-1)}P_{\bar{\varepsilon}_\mu}, \quad \pi_{\mu\nu} = \pi_\mu - \pi_\nu = rL_{\mu\nu} - (r-1)K_{\mu\nu}.$$

The operators  $K_{\mu\nu}$ ,  $L_{\mu\nu}$  and  $\pi_{\mu\nu}$  act on  $\mathcal{F}_{l,k}$  as scalars  $\langle \varepsilon_\mu - \varepsilon_\nu, k \rangle$ ,  $\langle \varepsilon_\mu - \varepsilon_\nu, l \rangle$  and  $\langle \varepsilon_\mu - \varepsilon_\nu, rl - (r-1)k \rangle$ , respectively. In what follows we often use the symbols

$$G_K = \prod_{0 \leq \mu < \nu \leq n-1} [K_{\mu\nu}], \quad G'_L = \prod_{0 \leq \mu < \nu \leq n-1} [L_{\mu\nu}]'.$$

For  $0 \leq \mu \leq n - 1$  define the type I vertex operator [12] by

$$\phi_\mu(v_0) = \oint_C \prod_{j=1}^\mu \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_1}(v_0) U_{-\alpha_1}(v_1) \cdots U_{-\alpha_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f(v_{j+1} - v_j, K_{j\mu}) \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} [K_{j\mu}]^{-1} \quad (4.5)$$

$$\begin{aligned} &= (-1)^\mu \oint_C \prod_{j=1}^\mu \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_\mu}(v_\mu) \cdots U_{-\alpha_1}(v_1) U_{\omega_1}(v_0) \\ &\times \prod_{j=0}^{\mu-1} f(v_j - v_{j+1}, 1 - K_{j\mu}) \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} [K_{j\mu}]^{-1}, \end{aligned} \quad (4.6)$$

where  $z_j = x^{2v_j}$ . Considering the factors  $f(v_{j+1} - v_j, K_{j\mu})$ 's together with the OPE formulae (B.3) and (B.5), the expression (4.5) has poles at  $z_j = x^{\pm(1+2kr)}z_{j-1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ). The integral contour  $C$  for  $z_j$ -integration should be chosen such that all integral variables lie in the common convergence domain; i.e. the contour  $C$  encircles the poles at  $z_j = x^{1+2kr}z_{j-1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ), but not the poles at  $z_j = x^{-1-2kr}z_{j-1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ).

Note that

$$\phi_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k+\bar{\varepsilon}_\mu}. \quad (4.7)$$

These type I vertex operators satisfy the following commutation relations on  $\mathcal{F}_{l,k}$ :

$$\phi_{\mu_1}(v_1)\phi_{\mu_2}(v_2) = \sum_{\varepsilon_{\mu_1+\mu_2}=\varepsilon_{\mu_1'}+\varepsilon_{\mu_2'}} W \left[ \begin{array}{c|c} a + \bar{\varepsilon}_{\mu_1} + \bar{\varepsilon}_{\mu_2} & a + \bar{\varepsilon}_{\mu_1'} \\ a + \bar{\varepsilon}_{\mu_2} & a \end{array} \middle| v_1 - v_2 \right] \phi_{\mu_2'}(v_2)\phi_{\mu_1'}(v_1). \quad (4.8)$$

We thus denote the operator  $\phi_\mu(v)$  by  $\Phi(v)_a^{a+\bar{\varepsilon}_\mu}$  on the bosonic Fock space  $\mathcal{F}_{l,a+\rho}$ .

Dual vertex operators are likewise defined as follows:

$$\begin{aligned} \phi_\mu^*(v) &= (-1)^{n-1-\mu} c_n^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_{n-1}}\left(v - \frac{n}{2}\right) U_{-\alpha_{n-1}}(v_{n-1}) \cdots U_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\quad \times \prod_{j=\mu+1}^{n-1} f(v_j - v_{j+1}, K_{\mu j}) \\ &= c_n^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots U_{-\alpha_{n-1}}(v_{n-1}) U_{\omega_{n-1}}\left(v - \frac{n}{2}\right) \\ &\quad \times \prod_{j=\mu+1}^{n-1} f(v_{j+1} - v_j, 1 - K_{\mu j}). \end{aligned} \tag{4.9}$$

Here  $v_n = v - \frac{n}{2}$ , and

$$c_n = x^{\frac{r-1}{r} \frac{n-1}{2n}} \frac{g_{n-1}(x^n)}{(x^2; x^{2r})_\infty^n (x^{2r}; x^{2r})_\infty^{2n-3}},$$

where  $g_{n-1}(z)$  is defined by (2.3). The integral contour for  $z_j$ -integration encircles the poles at  $z_j = x^{1+2kr} z_{j+1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ), but not the poles at  $z_j = x^{-1-2kr} z_{j+1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ), for  $\mu + 1 \leq j \leq n - 1$ . Note that

$$\phi_\mu^*(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k-\bar{\varepsilon}_\mu}. \tag{4.10}$$

The operators  $\phi_\mu(v)$  and  $\phi_\mu^*(v)$  are dual in the following sense [12]:

$$\sum_{\mu=0}^{n-1} \phi_\mu^*(v) \phi_\mu(v) = 1. \tag{4.11}$$

In [1] we obtained the free-field representation of  $\Lambda(u)_{\xi a}^{\xi a'}$  satisfying (3.27) for  $\xi' = \xi$ :

$$\begin{aligned} \Lambda(u)_{\xi a-\bar{\varepsilon}_v}^{\xi a-\bar{\varepsilon}_\mu} &= G_K \oint \prod_{j=\mu+1}^v \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots U_{-\alpha_v}(v_v) \\ &\quad \times \prod_{j=\mu}^{v-1} f(v_{j+1} - v_j, K_{jv}) G_K^{-1}, \end{aligned} \tag{4.12}$$

where  $v_\mu = u$  and  $\mu < v$ .

### 4.3. Type II vertex operators

Let us define the basic operators for  $j = 1, \dots, n - 1$

$$V_{-\alpha_j}(v) = \exp(-\beta_2(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j} \log z)) : \exp\left(-\sum_{m \neq 0} \frac{1}{m} (A_m^j - A_m^{j+1})(x^j z)^{-m}\right) :, \tag{4.13}$$

$$V_{\omega_j}(v) = \exp(\beta_2(\sqrt{-1}Q_{\omega_j} + P_{\omega_j} \log z)) : \exp\left(\sum_{m \neq 0} \frac{1}{m} \sum_{k=1}^j x^{(j-2k+1)m} A_m^k z^{-m}\right) :, \tag{4.14}$$

where  $\beta_2 = \sqrt{\frac{r}{r-1}}$  and  $z = x^{2v}$ , and

$$A_m^j = (-1)^m \frac{[rm]_x}{[(r-1)m]_x} B_m^j. \tag{4.15}$$

For some useful OPE formulae and commutation relations, see appendix B.

For  $0 \leq \mu \leq n-1$  define the type II vertex operator [13, 14]<sup>1</sup> by

$$\psi_\mu^*(v_0) = \oint_{C'} \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{\omega_1}(v_0) V_{-\alpha_1}(v_1) \cdots V_{-\alpha_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f^*(v_{j+1} - v_j, L_{j\mu}) \tag{4.16}$$

$$\begin{aligned} &= (-1)^\mu \oint_{C'} \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{-\alpha_\mu}(v_\mu) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v_0) \\ &\times \prod_{j=0}^{\mu-1} f^*(v_j - v_{j+1}, 1 - L_{j\mu}), \end{aligned} \tag{4.17}$$

where  $z_j = x^{2v_j}$ . Considering the factors  $f^*(v_{j+1} - v_j, L_{j\mu})$ 's together with the OPE formulae (B.9) and (B.11), the expression (4.16) has poles at  $z_j = x^{\pm(-1+2k(r-1))} z_{j-1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ). The integral contour  $C'$  for  $z_j$ -integration should be chosen such that  $C'$  encircles the poles at  $z_j = x^{-1+2k(r-1)} z_{j-1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ), but not the poles at  $z_j = x^{1-2k(r-1)} z_{j-1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ).

Note that

$$\psi_\mu^*(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l+\bar{\varepsilon}_\mu, k}. \tag{4.18}$$

These type II vertex operators satisfy the following commutation relations on  $\mathcal{F}_{l,k}$ :

$$\psi_{\mu_1}^*(v_1) \psi_{\mu_2}^*(v_2) = \sum_{\varepsilon_{\mu_1} + \varepsilon_{\mu_2} = \varepsilon_{\mu_1'} + \varepsilon_{\mu_2'}} W' \left[ \begin{matrix} \xi + \bar{\varepsilon}_{\mu_1} + \bar{\varepsilon}_{\mu_2} & \xi + \bar{\varepsilon}_{\mu_2} \\ \xi + \bar{\varepsilon}_{\mu_1'} & \xi \end{matrix} \middle| v_2 - v_1 \right] \psi_{\mu_2'}^*(v_2) \psi_{\mu_1'}^*(v_1). \tag{4.19}$$

We thus denote the operator  $\psi_\mu^*(v)$  by  $\Psi_\mu^*(v)_{\xi+\bar{\varepsilon}_\mu}$  on the bosonic Fock space  $\mathcal{F}_{\xi+\rho, k}$ .

Dual vertex operators are likewise defined as follows:

$$\begin{aligned} \psi_\mu(v) &= (-1)^{n-1-\mu} c_n'^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{\omega_{n-1}} \left( v - \frac{n}{2} \right) V_{-\alpha_{n-1}}(v_{n-1}) \cdots V_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\times \prod_{j=\mu+1}^{n-1} f^*(v_j - v_{j+1}, L_{\mu j}) \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} \frac{[1]'}{[L_{j\mu}]'} \\ &= c_n'^{-1} \oint \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} V_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots V_{-\alpha_{n-1}}(v_{n-1}) V_{\omega_{n-1}} \left( v - \frac{n}{2} \right) \\ &\times \prod_{j=\mu+1}^{n-1} f^*(v_{j+1} - v_j, 1 - L_{\mu j}) \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} \frac{[1]'}{[L_{j\mu}]'}. \end{aligned} \tag{4.20}$$

<sup>1</sup> To be precise, the integral contour for  $\psi_\mu^*(v_0)$  of [13] is different from that of [14]. The contour should be chosen in such a way that all integral variables lie in the convergence domain of the integral formula (4.16). In the present paper we adopt the contour of [14].

Here  $v_n = v - \frac{n}{2}$ , and

$$c'_n = x^{\frac{r(n-1)}{2(r-1)}} \left( \frac{(x^{2r}; x^{2r-2})_\infty}{(x^{2r-2}; x^{2r-2})_\infty} \right)^n g_{n-1}^*(x^n),$$

where  $g_{n-1}^*(z)$  is defined by (2.4). The integral contour for  $z_j$ -integration encircles the poles at  $z_j = x^{-1+2k(r-1)}z_{j+1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ), but not the poles at  $z_j = x^{1-2k(r-1)}z_{j+1}$  ( $k \in \mathbb{Z}_{\geq 0}$ ), for  $\mu + 1 \leq j \leq n - 1$ . Note that

$$\psi_\mu(v) : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l-\bar{\varepsilon}_\mu,k}. \tag{4.21}$$

The operators  $\psi_\mu(v)$  and  $\psi_\mu^*(v)$  are dual in the following sense [14]:

$$\psi_\mu(v)\psi_\nu^*(v') = \delta_{\mu\nu} \frac{1}{1 - z'/z} + (\text{regular terms at } v = v'). \tag{4.22}$$

For later convenience, we also introduce another type of basic operators:

$$W_{-\alpha_j}(v) = \exp(-\beta_0(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j} \log(-1)^r z)) : \exp \left( - \sum_{m \neq 0} \frac{1}{m} (O_m^j - O_m^{j+1})(x^j z)^{-m} \right) :, \tag{4.23}$$

where  $\beta_0 = \beta_1 + \beta_2 = \frac{1}{\sqrt{r(r-1)}}$ ,  $(-1)^r := \exp(\pi\sqrt{-1}r)$  and

$$O_m^j = \frac{[m]_x}{[(r-1)m]_x} B_m^j. \tag{4.24}$$

Concerning useful OPE formulae and commutation relations, see appendix B.

#### 4.4. Free-field realization of tail operators

Consider (3.30) for  $(\xi_a, \xi_d, \xi_c) = (\xi, \xi, \xi + \bar{\varepsilon}_{n-1})$ , and  $(a, a') \rightarrow (a + \bar{\varepsilon}_{n-2}, a + \bar{\varepsilon}_{n-1})$ :

$$\Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_{n-1}} \Lambda(u)_\xi^{a+\bar{\varepsilon}_{n-1}} = \sum_{\mu=0}^{n-1} L' \left[ \begin{matrix} \xi + \bar{\varepsilon}_{n-1} & \xi \\ \xi + \bar{\varepsilon}_\mu & \xi \end{matrix} \middle| u + \Delta u - v \right] \Lambda(v)_{\xi+\bar{\varepsilon}_\mu}^{\xi+\bar{\varepsilon}_{n-1} a+\bar{\varepsilon}_{n-1}} \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}. \tag{4.25}$$

This equation can be rewritten as follows:

$$\begin{aligned} & \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_{n-1}} \Lambda(u)_\xi^{a+\bar{\varepsilon}_{n-1}} - \Lambda(u)_{\xi+\bar{\varepsilon}_{n-1}}^{\xi+\bar{\varepsilon}_{n-1} a+\bar{\varepsilon}_{n-1}} \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_{n-1}} \\ &= \sum_{\mu=0}^{n-2} \frac{[u + \Delta u - v + \xi_\mu n - 1]'}{[u + \Delta u - v]'} \prod_{j \neq \mu} \frac{[\xi_{n-1} j + 1]'}{[\xi_\mu j + 1]'} \Lambda(v)_{\xi+\bar{\varepsilon}_\mu}^{\xi+\bar{\varepsilon}_{n-1} a+\bar{\varepsilon}_{n-1}} \Psi^*(v)_\xi^{\xi+\bar{\varepsilon}_\mu}. \end{aligned} \tag{4.26}$$

Since the tail operators on the LHS of (4.26) are diagonal components with respect to the ground state sectors, the free-field representation (4.12) can be used. Thus, we have

$$\begin{aligned} \text{LHS of (4.26)} &= (-1)^{n-1} G_K \oint_C \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} [V_{-\alpha_{n-1}}(v_{n-1}), U_{-\alpha_{n-1}}(v')] \\ &\quad \times V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) f(v' - u, K_{n-2n-1}) \\ &\quad \times \prod_{j=0}^{n-2} f^*(v_j - v_{j+1}, 1 - L_{jn-1}) G_K^{-1}, \end{aligned} \tag{4.27}$$

where  $z_j = x^{2v_j}$  and  $z' = x^{2v'}$ . From (B.29) the integral with respect to  $z_{n-1}$  of (4.27) can be evaluated by the residues at  $z_{n-1} = -x^{\pm 1}z'$ . Then the result is

$$\begin{aligned} \text{LHS}|_{\mathcal{F}_{\xi+\rho, a+\bar{\epsilon}_{n-2}+\rho}} &= \frac{(-1)^{n-1}}{x^{-1}-x} G_K \oint_C \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ &\quad \times \left( F\left(v' + \frac{r}{2}\right) W_{-\alpha_{n-1}}\left(v' + \frac{r}{2}\right) - F\left(v' - \frac{r}{2}\right) W_{-\alpha_{n-1}}\left(v' - \frac{r}{2}\right) \right) \\ &\quad \times V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \prod_{j=0}^{n-3} f^*(v_j - v_{j+1}, 1 - \xi_{jn-1}) G_K^{-1} \\ &= \frac{(-1)^n}{x^{-1}-x} G_K \left( \oint_{x^{-r}C} - \oint_{x^rC} \right) \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} F(v') W_{-\alpha_{n-1}}(v') \\ &\quad \times V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \prod_{j=0}^{n-3} f^*(v_j - v_{j+1}, 1 - \xi_{jn-1}) G_K^{-1}, \end{aligned} \quad (4.28)$$

where

$$F(v') = \frac{[v_{n-2} - v' + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} - \xi_{n-2n-1}]' [v' - u - \frac{r+1}{2} - a_{n-2n-1}]}{[v_{n-2} - v' + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}]' [v' - u - \frac{r+1}{2}]}. \quad (4.29)$$

The integral with respect to  $z'$  of (4.28) can be evaluated by the residues at  $z' = -x^r z_{n-2}$  and  $z' = x^{-r+1+2u}$ . The former residue vanishes because of (B.40)<sup>2</sup>. Thus, we have

$$\begin{aligned} (4.28) &= (-1)^n \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}\left(u - \frac{r-1}{2}\right) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \\ &\quad \times \prod_{j=0}^{n-2} f^*(v_j - v_{j+1}, 1 - \xi_{jn-1}) \frac{[a_{n-2n-1}]}{(x^{-1}-x)(x^{2r}; x^{2r})_{\infty}^3} \frac{G_{a+\bar{\epsilon}_{n-1}}}{G_{a+\bar{\epsilon}_{n-2}}}. \end{aligned} \quad (4.30)$$

In (4.30), we should read as  $v_{n-1} = u + \frac{\pi\sqrt{-1}}{2\epsilon}$ . Equating (4.30) and the RHS of (4.26) and using the identity (2.10), we find the free-field representation of the tail operator

$$\begin{aligned} \Lambda(v)_{\xi+\bar{\epsilon}_{\mu} a+\bar{\epsilon}_{n-2}}^{\xi+\bar{\epsilon}_{n-1} a+\bar{\epsilon}_{n-1}} &= \frac{(-1)^{n-\mu} [a_{n-2n-1}]}{(x^{-1}-x)(x^{2r}; x^{2r})_{\infty}^3} \frac{[\xi_{\mu n-1} - 1]'}{[1]'} G_K G_L^{-1} \\ &\quad \times \oint_{C'} \prod_{j=\mu+1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}\left(u - \frac{r-1}{2}\right) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\quad \times \prod_{j=\mu+1}^{n-2} f^*(v_j - v_{j+1}, L_{\mu j}) G_K^{-1} G_L', \end{aligned} \quad (4.31)$$

for  $0 \leq \mu \leq n-2$  with  $\Delta u = -\frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$  and  $v_{n-1} = u + \frac{\pi\sqrt{-1}}{2\epsilon}$ .

Let us return to equation (3.30) with  $\Delta u = -\frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$ . By taking an appropriate linear combination of (3.30), we have the following relation:

$$\sum_{\mu=0}^{n-1} A_{\mu} \Psi^*(v)_{\xi'-\bar{\epsilon}_{\mu} a}^{\xi'} \Lambda(u)_{\xi a}^{\xi'-\bar{\epsilon}_{\mu} a'} = B \Lambda(u)_{\xi+\bar{\epsilon}_0 a}^{\xi' a'} \Psi^*(v)_{\xi}^{\xi+\bar{\epsilon}_0}. \quad (4.32)$$

<sup>2</sup> When  $n = 2$  we use (B.41).

Here, the coefficients are

$$A_\mu = \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} \frac{1}{[\xi'_{\mu j}]'} \frac{[u - v - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon} + \bar{\xi}'_\mu - \bar{\xi}_0 + \frac{1}{n}]' [\bar{\xi}'_0 - \bar{\xi}_0 + \frac{1}{n}]'}{[u - v - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon} + \bar{\xi}'_0 - \bar{\xi}_0 + \frac{1}{n}]' [\bar{\xi}'_\mu - \bar{\xi}_0 + \frac{1}{n}]'},$$

$$B = \frac{[u - v - \frac{n-3}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}]'}{[u - v - \frac{n-3}{2} + \frac{\pi\sqrt{-1}}{2\epsilon} + \bar{\xi}'_0 - \bar{\xi}_0 + \frac{1}{n}]'} \prod_{j=1}^{n-1} \frac{[\xi'_{j0}]'}{[\bar{\xi}'_j - \bar{\xi}_0 + \frac{1}{n}]' [\xi_{0j} + 1]'}$$

Consider the product

$$V_{\omega_1}(v) V_{-\alpha_1}(v_1) \cdots V_{-\alpha_{n-2}}(v_{n-2}) W_{-\alpha_{n-2}} \left( u - \frac{r-1}{2} \right)$$

$$= : V_{\omega_1}(v) V_{-\alpha_1}(v_1) \cdots V_{-\alpha_{n-2}}(v_{n-2}) W_{-\alpha_{n-2}} \left( u - \frac{r-1}{2} \right) :$$

$$\times \prod_{j=1}^{n-2} z_{j-1}^{-\frac{r}{r-1}} \frac{(x^{2r-1} \frac{z_j}{z_{j-1}}; x^{2r-2})_\infty}{(x^{-1} \frac{z_j}{z_{j-1}}; x^{2r-2})_\infty} \cdot z_{n-2}^{-\frac{1}{r-1}} \frac{(-x \frac{x^{2u}}{z_{n-2}}; x^{2r-2})_\infty}{(-x^{-1} \frac{x^{2u}}{z_{n-2}}; x^{2r-2})_\infty}. \tag{4.33}$$

The convergence domain of (4.33) is that  $x^{-1}|z_j| < |z_{j-1}|$  ( $1 \leq j \leq n-2$ ) and  $| -x^{2u-1} | < |z_{n-2}|$ . Thus, each term of the LHS of (4.32) has a pole at  $z = -x^{1-n}x^{2u}$  ( $v = u - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$ ) because pinching occurs at the pole. On the other hand, the RHS of (4.32) does not have such a pole. Hence, the singularities at  $v = u - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$  on the RHS of (4.32) cancel each other:

$$\sum_{\mu=0}^{n-1} \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} \frac{1}{[\xi'_{\mu j}]'} \Psi^* \left( u - \frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon} \right)_{\xi' - \bar{\epsilon}_\mu}^{\xi'} \Lambda(u)_{\xi a}^{\xi' - \bar{\epsilon}_\mu a'} = O(1). \tag{4.34}$$

From (4.34) and (2.10) we find the representation

$$\Lambda(u)_{\xi a}^{\xi' - \bar{\epsilon}_\mu a'} = \oint_{C'} \prod_{j=\mu+1}^{n-1} f^*(v_j - v_{j+1}, L_{j\mu}) \frac{dz_j}{2\pi\sqrt{-1}z_j}$$

$$\times V_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots V_{-\alpha_{n-1}}(v_{n-1}) \cdot \Lambda(u)_{\xi a}^{\xi' - \bar{\epsilon}_{n-1} a'}, \tag{4.35}$$

where  $v_n = u + \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$ .

In a similar way to derive (4.32) from (3.30), we can derive the following relation from (3.27):

$$\sum_{\mu=0}^{n-1} \Lambda(u)_{\xi a + \bar{\epsilon}_\mu}^{\xi' a'} \Phi(v)_{a + \bar{\epsilon}_\mu}^{a + \bar{\epsilon}_\mu} \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} \frac{[a_{\mu j} + 1]}{[a_{\mu j}]} \left[ u - v + \bar{a}'_v - \bar{a}_\mu + \frac{1}{n} \right] \prod_{\substack{j=0 \\ j \neq v}}^{n-1} \left[ \bar{a}'_j - \bar{a}_\mu + \frac{1}{n} \right]$$

$$= [u - v + 1] \prod_{\substack{j=0 \\ j \neq v}}^{n-1} [a'_{v j}] \Phi(v)_{a' - \bar{\epsilon}_v}^{a' - \bar{\epsilon}_v} \Lambda(u)_{\xi a}^{\xi' a' - \bar{\epsilon}_v}. \tag{4.36}$$

Let  $v = u + 1$  and take the sum over  $0 \leq v \leq n-1$ . Then we have

$$\sum_{\mu=0}^{n-1} A_\mu(a, a') \Lambda(u)_{\xi a + \bar{\epsilon}_\mu}^{\xi' a'} \Phi(u+1)_a^{a + \bar{\epsilon}_\mu} \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} \frac{[(a + \bar{\epsilon}_\mu)_{\mu j}]}{[a_{\mu j}]} = 0, \tag{4.37}$$



where

$$A_\mu(a, a') = \sum_{v=0}^{n-1} \prod_{j=0}^{n-1} [(a' - \bar{\epsilon}_v)_j - \bar{a}_\mu].$$

From (4.37) and (2.9), we obtain the expression

$$\begin{aligned} \Lambda(u)_{\xi a+\bar{\epsilon}_\mu}^{\xi' a'} &= \Lambda(u)_{\xi a+\bar{\epsilon}_{n-1}}^{\xi' a'} (-1)^{n-1-\mu} G_K \oint_C \prod_{j=\mu+1}^{n-1} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{n-1}}(v_{n-1}) \cdots U_{-\alpha_{\mu+1}}(v_{\mu+1}) \\ &\times \prod_{j=\mu+1}^{n-1} f(v_j - v_{j+1}, K_\mu j) G_K^{-1} \frac{A_{n-1}(a, a')}{A_\mu(a, a')}, \end{aligned} \tag{4.38}$$

where  $v_n = u - \frac{n-2}{2}$ .

Combining equations (4.31), (4.35) and (4.38), we can construct a free-field representation of any  $\Lambda(u)_{\xi a}^{\xi' a'}$ , in principle.

#### 4.5. Form factors

Form factors of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model are defined as matrix elements of some local operators. Consider the local operator

$$\mathcal{O} = E_{\mu_1 \mu'_1}^{(1)} \cdots E_{\mu_N \mu'_N}^{(N)}, \tag{4.39}$$

where  $E_{\mu_j \mu'_j}^{(j)}$  is the matrix unit on the  $j$ th site. The free-field representation of  $\mathcal{O}$  is given by

$$\hat{\mathcal{O}} = \Phi_{\mu_1}^*(u_1) \cdots \Phi_{\mu_N}^*(u_N) \Phi^{\mu'_N}(u_N) \cdots \Phi^{\mu'_1}(u_1). \tag{4.40}$$

The corresponding form factors with  $m$  ‘charged’ particles are given by

$$F_m^{(i)}(\mathcal{O}; v_1, \dots, v_m)_{v_1 \cdots v_m} = \frac{1}{\chi^{(i)}} \text{Tr}_{\mathcal{H}^{(i)}} (\Psi_{v_1}^*(v_1) \cdots \Psi_{v_m}^*(v_m) \hat{\mathcal{O}} \rho^{(i)}), \tag{4.41}$$

where

$$\chi^{(i)} = \text{Tr}_{\mathcal{H}^{(i)}} \rho^{(i)} = \frac{(x^{2n}; x^{2n})_\infty}{(x^2; x^2)_\infty} \tag{4.42}$$

and  $m \equiv 0 \pmod{n}$ . Note that the local operator (4.39) commutes with the type II vertex operators due to of (4.40) and (3.6).

By using (3.18), (3.28) and (3.19), we can rewrite (4.41) as follows:

$$\begin{aligned} F_m^{(i)}(\mathcal{O}; v_1, \dots, v_m)_{v_1 \cdots v_m} &= \frac{1}{\chi^{(i)}} \sum_{\xi_1, \dots, \xi_m} t_{v_1}^* \left( v_1 - u + \frac{n-1}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} \right)_{\xi_1}^{\xi_1} \cdots \\ &\times t_{v_m}^* \left( v_m - u + \frac{n-1}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} \right)_{\xi_m}^{\xi_m} \\ &\times \sum_{\substack{k \equiv l + \omega_l \\ \pmod{Q}}} \sum_{\substack{a_1 \cdots a_N \\ a'_1 \cdots a'_N}} t_{\mu_1}^*(u_1 - u)_{a_1}^{a_1} \cdots t_{\mu_N}^*(u_N - u)_{a_N}^{a_N} t^{\mu'_N}(u_N - u)_{a'_N}^{a'_N} \cdots t^{\mu'_1}(u_1 - u)_{a'_1}^{a'_1} \\ &\times \text{Tr}_{\mathcal{H}_{l,k}^{(i)}} \left( \Psi^*(v_1)_{\xi_1}^{\xi_1} \cdots \Psi^*(v_m)_{\xi_m}^{\xi_m} \Phi^*(u_1)_{a_1}^{a_1} \cdots \Phi^*(u_N)_{a_N}^{a_N} \Phi(u_N)_{a'_N}^{a'_N} \right. \\ &\left. \cdots \Phi(u_1)_{a'_1}^{a'_1} \Lambda(u)_{\xi a}^{\xi_m a'_1} \frac{\rho_{l,k}^{(i)}}{b_l} \right), \end{aligned} \tag{4.43}$$

where  $k = a + \rho, l = \xi + \rho$  and

$$b_l = \left( \frac{(x^{2r}; x^{2r})_\infty}{(x^{2r-2}; x^{2r-2})_\infty} \right)^{(n-1)(n-2)/2} G'_\xi. \tag{4.44}$$

Free-field representations of the tail operators  $\Lambda$ 's have been constructed in the present paper, with the exception of all the other operators  $\Phi$ 's,  $\Phi^*$ 's and  $\Psi^*$ 's in (4.43), were given in [1, 12, 14]. In principle, integral formulae can be therefore obtained for form factors of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model.

**5. Concluding remarks**

In this paper we present a vertex operator approach for form factors of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model. For that purpose we constructed the free-field representations of the tail operators  $\Lambda_{\xi a}^{\xi' a'}$ , the nonlocal operators which relate the physical quantities of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and the  $A_{n-1}^{(1)}$ -model. As a result, we can obtain the integral formulae for form factors of the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model, in principle.

Our approach is based on some assumptions. We assumed that the vertex operator algebra (3.18)–(3.20) and (3.28), (3.29) correctly describes the intertwining relation between the  $(\mathbb{Z}/n\mathbb{Z})$ -symmetric model and the  $A_{n-1}^{(1)}$ -model. We also assumed that the free-field representations (4.31), (4.35), (4.38) provide relevant representations of the vertex operator algebra. As a consistency check of our bosonization scheme, it is thus important to derive closed expressions for form factors of some simple local operators by performing the integrals on (4.43). We wish to address the problem in a separate paper.

Before ending the present paper, we would like to add one more point. In order to find the free-field representations of the tail operators (4.31), we used the correct commutation relation (B.29). In our previous paper [14] we proved (3.14) by using the commutativity of  $U_{-\alpha_j}(v)$  and  $V_{-\alpha_j}(v')$ , instead of (B.29). In appendix C we thus prove (3.14) on the basis of (B.29).

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**Appendix A. Definitions of the models concerned**

*A.1. Belavin's vertex model*

In the original papers [2, 3], the  $R$ -matrix in the disordered phase is given. For the purpose of this paper, we need the following  $R$ -matrix:

$$R(v) = \frac{[1]}{[1-v]} r_1(v) \bar{R}(v),$$

$$\bar{R}(v)_{jl}^{ik} = \frac{h(v) \vartheta \left[ \frac{1}{2} + \frac{k-i}{n} \right] \left( \frac{1-v}{nr}; \frac{\pi\sqrt{-1}}{ner} \right)}{\vartheta \left[ \frac{1}{2} + \frac{j-k}{n} \right] \left( \frac{v}{nr}; \frac{\pi\sqrt{-1}}{ner} \right) \vartheta \left[ \frac{1}{2} + \frac{j-i}{n} \right] \left( \frac{1}{nr}; \frac{\pi\sqrt{-1}}{ner} \right)} \delta_{j+l \pmod n}^{i+k}, \tag{A.1}$$

where  $r_1(v)$  is defined by (2.3), and

$$h(v) = \prod_{j=0}^{n-1} \vartheta \left[ \frac{1}{2} + \frac{j}{n} \right] \left( \frac{v}{nr}; \frac{\pi\sqrt{-1}}{ner} \right) / \prod_{j=1}^{n-1} \vartheta \left[ \frac{1}{2} + \frac{j}{n} \right] \left( 0; \frac{\pi\sqrt{-1}}{ner} \right).$$

We assume that the parameters  $v, \epsilon$  and  $r$  lie in the so-called principal regime:

$$\epsilon > 0, \quad r > n - 1, \quad 0 < v < 1. \tag{A.2}$$

Note that the weights (A.1) reproduce those of the eight-vertex model in the principal regime when  $n = 2$  [5].

A.2. The weight lattice and the root lattice of  $A_{n-1}^{(1)}$

Let  $V = \mathbb{C}^n$  and  $\{\epsilon_\mu\}_{0 \leq \mu \leq n-1}$  be the standard orthonormal basis as before. The weight lattice of  $A_{n-1}^{(1)}$  is defined as follows:

$$P = \bigoplus_{\mu=0}^{n-1} \mathbb{Z}\bar{\epsilon}_\mu, \tag{A.3}$$

where

$$\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon, \quad \epsilon = \frac{1}{n} \sum_{\mu=0}^{n-1} \epsilon_\mu.$$

We denote the fundamental weights by  $\omega_\mu$  ( $1 \leq \mu \leq n - 1$ ):

$$\omega_\mu = \sum_{v=0}^{\mu-1} \bar{\epsilon}_v,$$

and also denote the simple roots by  $\alpha_\mu$  ( $1 \leq \mu \leq n - 1$ ):

$$\alpha_\mu = \epsilon_{\mu-1} - \epsilon_\mu = \bar{\epsilon}_{\mu-1} - \bar{\epsilon}_\mu.$$

The root lattice of  $A_{n-1}^{(1)}$  is defined as follows:

$$Q = \bigoplus_{\mu=1}^{n-1} \mathbb{Z}\alpha_\mu. \tag{A.4}$$

For  $a \in P$  we set

$$a_{\mu v} = \bar{a}_\mu - \bar{a}_v, \quad \bar{a}_\mu = \langle a + \rho, \epsilon_\mu \rangle = \langle a + \rho, \bar{\epsilon}_\mu \rangle, \quad \rho = \sum_{\mu=1}^{n-1} \omega_\mu. \tag{A.5}$$

In this paper we not only admit the case  $a \in P$ , but also the case  $a \in \mathfrak{h}^* := \mathbb{C}\omega_0 \oplus \mathbb{C}\omega_1 \oplus \dots \oplus \mathbb{C}\omega_{n-1}$ . For  $r > n - 1$ , let  $\sum_{\mu=0}^{n-1} k^\mu = r$ , where  $a + \rho = \sum_{\mu=0}^{n-1} k^\mu \omega_\mu$ ; then we denote  $a \in \mathfrak{h}_{r-n}^*$ .

A.3. The  $A_{n-1}^{(1)}$  face model

An ordered pair  $(a, b) \in \mathfrak{h}_{r-n}^{*2}$  is called *admissible* if  $b = a + \bar{\epsilon}_\mu$ , for a certain  $\mu$  ( $0 \leq \mu \leq n - 1$ ). Non-zero Boltzmann weights are parametrized in terms of the elliptic theta function of the spectral parameter  $v$  as follows:

$$W \left[ \begin{matrix} a + 2\bar{\epsilon}_\mu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu & a \end{matrix} \middle| v \right] = r_1(v), \tag{A.6}$$

$$W \left[ \begin{matrix} a + \bar{\epsilon}_\mu + \bar{\epsilon}_v & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_v & a \end{matrix} \middle| v \right] = r_1(v) \frac{[v][a_{\mu v} + 1]}{[1 - v][a_{\mu v}]} \quad (\mu \neq v), \tag{A.7}$$

$$W \left[ \begin{matrix} a + \bar{\epsilon}_\mu + \bar{\epsilon}_v & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu & a \end{matrix} \middle| v \right] = r_1(v) \frac{[1][v + a_{\mu v}]}{[1 - v][a_{\mu v}]} \quad (\mu \neq v), \tag{A.8}$$

where  $r_1(v)$  is defined by (2.3). In this paper we consider the so-called Regime III in the model, i.e.  $0 < v < 1$ .

**Appendix B. OPE formulae and commutation relations**

In this appendix we list some useful formulae for the basic bosons. In what follows we denote  $z = x^{2v}$ ,  $z' = x^{2v'}$ .

First, useful OPE formulae are

$$U_{\omega_1}(v)U_{\omega_j}(v') = z^{\frac{r-1}{r} \frac{n-j}{n}} g_j(z'/z) : U_{\omega_1}(v)U_{\omega_j}(v') :, \tag{B.1}$$

$$U_{\omega_j}(v)U_{\omega_1}(v') = z^{\frac{r-1}{r} \frac{n-j}{n}} g_j(z'/z) : U_{\omega_j}(v)U_{\omega_1}(v') :, \tag{B.2}$$

$$U_{\omega_j}(v)U_{-\alpha_j}(v') = z^{-\frac{r-1}{r}} \frac{(x^{2r-1}z'/z; x^{2r})_{\infty}}{(xz'/z; x^{2r})_{\infty}} : U_{\omega_j}(v)U_{-\alpha_j}(v') :, \tag{B.3}$$

$$U_{-\alpha_j}(v)U_{\omega_j}(v') = z^{-\frac{r-1}{r}} \frac{(x^{2r-1}z'/z; x^{2r})_{\infty}}{(xz'/z; x^{2r})_{\infty}} : U_{-\alpha_j}(v)U_{\omega_j}(v') :, \tag{B.4}$$

$$U_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') = z^{-\frac{r-1}{r}} \frac{(x^{2r-1}z'/z; x^{2r})_{\infty}}{(xz'/z; x^{2r})_{\infty}} : U_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') :, \tag{B.5}$$

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = z^{\frac{2(r-1)}{r}} \left(1 - \frac{z'}{z}\right) \frac{(x^2z'/z; x^{2r})_{\infty}}{(x^{2r-2}z'/z; x^{2r})_{\infty}} : U_{-\alpha_j}(v)U_{-\alpha_j}(v') :, \tag{B.6}$$

$$V_{\omega_1}(v)V_{\omega_j}(v') = z^{\frac{r}{r-1} \frac{n-j}{n}} g_j^*(z'/z) : V_{\omega_1}(v)V_{\omega_j}(v') :, \tag{B.7}$$

$$V_{\omega_j}(v)V_{\omega_1}(v') = z^{\frac{r}{r-1} \frac{n-j}{n}} g_j^*(z'/z) : V_{\omega_j}(v)V_{\omega_1}(v') :, \tag{B.8}$$

$$V_{\omega_j}(v)V_{-\alpha_j}(v') = z^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z; x^{2r-2})_{\infty}}{(x^{-1}z'/z; x^{2r-2})_{\infty}} : V_{\omega_j}(v)V_{-\alpha_j}(v') :, \tag{B.9}$$

$$V_{-\alpha_j}(v)V_{\omega_j}(v') = z^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z; x^{2r-2})_{\infty}}{(x^{-1}z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_j}(v)V_{\omega_j}(v') :, \tag{B.10}$$

$$V_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') = z^{-\frac{r}{r-1}} \frac{(x^{2r-1}z'/z; x^{2r-2})_{\infty}}{(x^{-1}z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') :, \tag{B.11}$$

$$V_{-\alpha_j}(v)V_{-\alpha_j}(v') = z^{\frac{2r}{r-1}} \left(1 - \frac{z'}{z}\right) \frac{(x^{-2}z'/z; x^{2r-2})_{\infty}}{(x^{2r}z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_j}(v)V_{-\alpha_j}(v') :, \tag{B.12}$$

$$V_{\omega_j}(v)U_{\omega_j}(v') = z^{-\frac{j(n-j)}{n}} \rho_j(z'/z) : V_{\omega_1}(v)U_{\omega_j}(v') :, \tag{B.13}$$

$$U_{\omega_j}(v)V_{\omega_j}(v') = z^{-\frac{j(n-j)}{n}} \rho_j(z'/z) : U_{\omega_j}(v)V_{\omega_j}(v') :, \tag{B.14}$$

$$V_{\omega_j}(v)U_{-\alpha_j}(v') = z \left(1 + \frac{z'}{z}\right) : V_{\omega_j}(v)U_{-\alpha_j}(v') := U_{-\alpha_j}(v')V_{\omega_j}(v), \tag{B.15}$$

$$U_{\omega_j}(v)V_{-\alpha_j}(v') = z \left(1 + \frac{z'}{z}\right) : U_{\omega_j}(v)V_{-\alpha_j}(v') := V_{-\alpha_j}(v')U_{\omega_j}(v), \tag{B.16}$$

$$V_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') = z \left(1 + \frac{z'}{z}\right) : V_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') := U_{-\alpha_{j\pm 1}}(v')V_{-\alpha_j}(v), \tag{B.17}$$

$$V_{-\alpha_j}(v)U_{-\alpha_j}(v') = \frac{V_{-\alpha_j}(v)U_{-\alpha_j}(v')}{z^2(1 + \frac{xz'}{z})(1 + \frac{x^{-1}z'}{z})}, \tag{B.18}$$

$$U_{-\alpha_j}(v)V_{-\alpha_j}(v') = \frac{U_{-\alpha_j}(v)V_{-\alpha_j}(v')}{z^2(1 + \frac{xz'}{z})(1 + \frac{x^{-1}z'}{z})}, \tag{B.19}$$

where  $g_j(z)$ ,  $g_j^*(z)$  and  $\rho_j(z)$  are defined by (2.3), (2.4) and (2.5). From these, we obtain the following commutation relations:

$$U_{\omega_1}(v)U_{\omega_j}(v') = r_j(v - v')U_{\omega_j}(v')U_{\omega_1}(v), \tag{B.20}$$

$$U_{-\alpha_j}(v)U_{\omega_j}(v') = -f(v - v', 0)U_{\omega_j}(v')U_{-\alpha_j}(v), \tag{B.21}$$

$$U_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') = -f(v - v', 0)U_{-\alpha_{j\pm 1}}(v')U_{-\alpha_j}(v), \tag{B.22}$$

$$U_{-\alpha_j}(v)U_{-\alpha_j}(v') = h(v - v')U_{-\alpha_j}(v')U_{-\alpha_j}(v), \tag{B.23}$$

$$V_{\omega_1}(v)V_{\omega_j}(v') = r_j^*(v - v')V_{\omega_j}(v')V_{\omega_1}(v), \tag{B.24}$$

$$V_{-\alpha_j}(v)V_{\omega_j}(v') = -f^*(v - v', 0)V_{\omega_j}(v')V_{-\alpha_j}(v), \tag{B.25}$$

$$V_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') = -f^*(v - v', 0)V_{-\alpha_{j\pm 1}}(v')V_{-\alpha_j}(v), \tag{B.26}$$

$$V_{-\alpha_j}(v)V_{-\alpha_j}(v') = h^*(v - v')V_{-\alpha_j}(v')V_{-\alpha_j}(v), \tag{B.27}$$

$$U_{\omega_j}(v)V_{\omega_j}(v') = \chi_j(v - v')V_{\omega_j}(v')U_{\omega_j}(v), \tag{B.28}$$

$$[V_{-\alpha_j}(v), U_{-\alpha_j}(v')] = \frac{\delta\left(\frac{z}{-xz'}\right) - \delta\left(\frac{z'}{-xz}\right)}{(x^{-1} - x)zz'} : V_{-\alpha_j}(v)U_{-\alpha_j}(v') :, \tag{B.29}$$

where  $r_j(v)$ ,  $r_j^*(v)$ ,  $\chi_j(v)$ ,  $f(v, w)$ ,  $h(v)$ ,  $f^*(v, w)$  and  $h^*(v)$  are defined by (2.3), (2.4), (2.5), (2.7) and (2.8), and the  $\delta$ -function is defined by the following formal power series:

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

The commutation relation (B.29) can be derived from (B.18), (B.19) and the identity

$$\frac{1}{z^2\left(1 + \frac{xz'}{z}\right)\left(1 + \frac{x^{-1}z'}{z}\right)} - \frac{1}{z'^2\left(1 + \frac{xz}{z'}\right)\left(1 + \frac{x^{-1}z}{z'}\right)} = \frac{\delta\left(\frac{z}{-xz'}\right) - \delta\left(\frac{z'}{-xz}\right)}{(x^{-1} - x)zz'}.$$

The relation (B.29) can be practically understood as follows. Let us compare the integrals

$$\oint \frac{dz}{2\pi\sqrt{-1}} V_{-\alpha_j}(v)U_{-\alpha_j}(v')F(v, v'), \tag{B.30}$$

and

$$\oint \frac{dz}{2\pi\sqrt{-1}} U_{-\alpha_j}(v')V_{-\alpha_j}(v)F(v, v'), \tag{B.31}$$

where  $F(u, v)$  is an appropriate function. Note that the normal order product expansion (B.18) is valid for  $|z| > |-x^{\pm 1}z'|$  while (B.19) is valid for  $|z'| > |-x^{\pm 1}z|$ . Thus, the integral contour of (B.30) encircles the poles  $-x^{\pm 1}z'$ , but that of (B.31) does not encircle them. The difference between (B.30) and (B.31) can be therefore evaluated by the residues at  $z = -x^{\pm 1}z'$ .

Finally, we list the OPE formulae for  $W_{-\alpha_j}(v)$  and other basic operators:

$$W_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') = -(-z)^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_{\infty}}{(-x^{r-2} z'/z; x^{2r-2})_{\infty}} : W_{-\alpha_j}(v)V_{-\alpha_{j\pm 1}}(v') :, \tag{B.32}$$

$$V_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}(v') = z^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_{\infty}}{(-x^{r-2} z'/z; x^{2r-2})_{\infty}} : V_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}(v') :, \tag{B.33}$$

$$V_{\omega_j}(v)W_{-\alpha_j}(v') = z^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_{\infty}}{(-x^{r-2} z'/z; x^{2r-2})_{\infty}} : V_{\omega_j}(v)W_{-\alpha_j}(v') :, \tag{B.34}$$

$$W_{-\alpha_j}(v)V_{\omega_j}(v') = -(-z)^{-\frac{1}{r-1}} \frac{(-x^r z'/z; x^{2r-2})_\infty}{(-x^{r-2} z'/z; x^{2r-2})_\infty} : W_{-\alpha_j}(v)V_{\omega_j}(v') :, \tag{B.35}$$

$$U_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}(v') = z^{\frac{1}{r}} \frac{(x^{r-1} z'/z; x^{2r})_\infty}{(x^{r+1} z'/z; x^{2r})_\infty} : U_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j}(v') :, \tag{B.36}$$

$$W_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') = -z^{\frac{1}{r}} \frac{(x^{r-1} z'/z; x^{2r})_\infty}{(x^{r+1} z'/z; x^{2r})_\infty} : W_{-\alpha_j}(v)U_{-\alpha_{j\pm 1}}(v') :, \tag{B.37}$$

$$U_{\omega_j}(v)W_{-\alpha_j}(v') = z^{\frac{1}{r}} \frac{(x^{r-1} z'/z; x^{2r})_\infty}{(x^{r+1} z'/z; x^{2r})_\infty} : U_{\omega_j}(v)W_{-\alpha_j}(v') :, \tag{B.38}$$

$$W_{-\alpha_j}(v)U_{\omega_j}(v') = -z^{\frac{1}{r}} \frac{(x^{r-1} z'/z; x^{2r})_\infty}{(x^{r+1} z'/z; x^{2r})_\infty} : W_{-\alpha_j}(v)U_{\omega_j}(v') :. \tag{B.39}$$

From these, we obtain

$$W_{-\alpha_j} \left( v + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} \right) V_{-\alpha_{j\pm 1}}(v) = 0 = V_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j} \left( v - \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} \right), \tag{B.40}$$

$$W_{-\alpha_j} \left( v + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} \right) V_{\omega_j}(v) = 0 = V_{\omega_j}(v)W_{-\alpha_j} \left( v - \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} \right), \tag{B.41}$$

$$U_{-\alpha_{j\pm 1}}(v)W_{-\alpha_j} \left( v - \frac{r-1}{2} \right) = 0 = W_{-\alpha_j} \left( v + \frac{r-1}{2} \right) U_{-\alpha_{j\pm 1}}(v), \tag{B.42}$$

$$U_{\omega_j}(v)W_{-\alpha_j} \left( v - \frac{r-1}{2} \right) = 0 = W_{-\alpha_j} \left( v + \frac{r-1}{2} \right) U_{\omega_j}(v). \tag{B.43}$$

### Appendix C. Commutation relations of $\Phi(u)_a^{a'}$ and $\Psi^*(v)_\xi^{\xi'}$

In this appendix, we give a remark on the commutation relation (3.14). In [14] we proved (3.14) on the assumption of the commutativity of  $U_{-\alpha_j}(v)$  and  $V_{-\alpha_j}(v')$ . From (B.29), however,  $U_{-\alpha_j}(v)$  and  $V_{-\alpha_j}(v')$  commute at all points but at  $v' = v \pm \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$ . Nevertheless, (3.14) holds, which we will briefly show in this appendix.

Let  $a' - a = \bar{\epsilon}_\mu$  and  $\xi' - \xi = \bar{\epsilon}_\nu$  on (3.14). We assume that  $\mu \leq \nu$ . (The case  $\mu > \nu$  can be similarly proved.) When  $\mu = 0$ , (3.14) follows from (B.15)–(B.17) and (B.28). When  $\mu = 1$ , the difference of the both sides of (3.14) can be calculated as follows:

$$\begin{aligned} & \Phi(u)_a^{a+\bar{\epsilon}_1} \Psi^*(v)_\xi^{\xi+\bar{\epsilon}_\nu} - \chi(u-v) \Psi^*(v)_\xi^{\xi+\bar{\epsilon}_\nu} \Phi(u)_a^{a+\bar{\epsilon}_1} \\ &= U_{\omega_1}(u)V_{\omega_1}(v) \oint_C \frac{dz'_1}{2\pi\sqrt{-1}z'_1} \oint_{C'} \prod_{j=1}^v \frac{dz_j}{2\pi\sqrt{-1}z_j} [U_{-\alpha_1}(u_1), V_{-\alpha_1}(v_1)] \\ & \quad \times V_{-\alpha_2}(v_2) \cdots V_{-\alpha_\nu}(v_\nu) f(u_1 - u, K_{01}) \prod_{\substack{j=0 \\ j \neq 1}}^{n-1} [K_{j1}]^{-1} \prod_{j=0}^{v-1} f^*(v_{j+1} - v_j, L_{j\nu}), \end{aligned} \tag{C.1}$$

where  $z_j = x^{2v_j}$  and  $z'_j = x^{2u_j}$ . From (B.29) the integral with respect to  $z_1$  of (C.1) can be evaluated by the residues at  $z_1 = -x^{\pm 1}z'_1$ . Repeating similar calculations performed in

section 4.4, the RHS of (C.1) can be rewritten as a total difference of such a form

$$U_{\omega_1}(u)V_{\omega_1}(v) \left( \oint_{x^{-r}C} - \oint_{x^rC} \right) \frac{dz'_1}{2\pi\sqrt{-1}z'_1} \oint_{C'} \prod_{j=2}^v \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ \times W_{-\alpha_1}(u_1)V_{-\alpha_2}(v_2) \cdots V_{-\alpha_v}(v_v)G(u_1), \tag{C.2}$$

where

$$G\left(u_1 + \frac{r}{2}\right) = \frac{1}{x^{-1} - x} f(u_1 - u, a_{01}) \prod_{\substack{j=0 \\ j \neq 1}}^{n-1} [a_{j1}]^{-1} \prod_{j=0}^{v-1} f^*(v_{j+1} - v_j, \xi_{jv}) \Big|_{v_1 = u_1 + \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}}.$$

In the present case, there are at most three poles at  $u_1 = u - \frac{r-1}{2}, v - \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}, v_2 + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}$ , inside the contour for  $z'_1$ -integration. The residues at those three points vanish because of (B.43), (B.41) and (B.40), respectively. Therefore, we have

$$\Phi(u)_a^{a+\bar{\epsilon}_1} \Psi^*(v)_\xi^{\xi+\bar{\epsilon}_v} - \chi(u-v) \Psi^*(v)_\xi^{\xi+\bar{\epsilon}_v} \Phi(v)_a^{a+\bar{\epsilon}_1} = 0.$$

When  $\mu \geq 2$ , the difference of the both sides of (3.14) can be calculated as follows:

$$\Phi(u)_a^{a+\bar{\epsilon}_\mu} \Psi^*(v)_\xi^{\xi+\bar{\epsilon}_v} - \chi(u-v) \Psi^*(v)_\xi^{\xi+\bar{\epsilon}_v} \Phi(v)_a^{a+\bar{\epsilon}_\mu} \\ = \sum_{\lambda=1}^{\mu} \oint_C \prod_{j=1}^{\mu} \frac{dz'_j}{2\pi\sqrt{-1}z'_j} \oint_{C'} \prod_{j'=1}^v \frac{dz_{j'}}{2\pi\sqrt{-1}z_{j'}} X_\lambda, \tag{C.3}$$

where

$$X_\lambda = U_{\omega_1}(u)V_{\omega_1}(v)V_{-\alpha_1}(v_1)U_{-\alpha_1}(v_1) \cdots V_{-\alpha_{\lambda-1}}(v_{\lambda-1})U_{-\alpha_{\lambda-1}}(u_{\lambda-1})[U_{-\alpha_\lambda}(u_\lambda), V_{-\alpha_\lambda}(v_\lambda)] \\ \times U_{-\alpha_{\lambda+1}}(u_{\lambda+1})V_{-\alpha_{\lambda+1}}(v_{\lambda+1}) \cdots U_{-\alpha_\mu}(u_\mu)V_{-\alpha_\mu}(v_\mu) \cdots V_{-\alpha_v}(v_v) \\ \times \prod_{j=0}^{\mu-1} f(u_{j+1} - u_j, K_{j\mu}) \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} [K_{j\mu}]^{-1} \prod_{j'=0}^{v-1} f^*(v_{j'+1} - v_{j'}, L_{j'v}). \tag{C.4}$$

From (B.29), the integral with respect to  $z_\lambda$  of  $X_\lambda$  can be evaluated by the residues at  $z_\lambda = -x^{\pm 1}z'_\lambda$ . Similarly to (C.2), the result can be rewritten as a total difference of such a form

$$\oint_C \prod_{j=1}^{\mu} \frac{dz'_j}{2\pi\sqrt{-1}z'_j} \oint_{C'} \prod_{j'=1}^v \frac{dz_{j'}}{2\pi\sqrt{-1}z_{j'}} X_\lambda = U_{\omega_1}(u)V_{\omega_1}(v) \left( \oint_{x^{-r}C} - \oint_{x^rC} \right) \frac{dz'_\lambda}{2\pi\sqrt{-1}z'_\lambda} \\ \times \oint_C \prod_{\substack{j=1 \\ j \neq \lambda}}^{\mu} \frac{dz'_j}{2\pi\sqrt{-1}z'_j} \oint_{C'} \prod_{\substack{j'=1 \\ j' \neq \lambda}}^v \frac{dz_{j'}}{2\pi\sqrt{-1}z_{j'}} \\ \times V_{-\alpha_1}(v_1)U_{-\alpha_1}(v_1) \cdots V_{-\alpha_{\lambda-1}}(v_{\lambda-1})U_{-\alpha_{\lambda-1}}(u_{\lambda-1}) \\ \times W_{-\alpha_\lambda}(u_\lambda)U_{-\alpha_{\lambda+1}}(u_{\lambda+1})V_{-\alpha_{\lambda+1}}(v_{\lambda+1}) \cdots U_{-\alpha_\mu}(u_\mu) \\ \times V_{-\alpha_\mu}(v_\mu) \cdots V_{-\alpha_v}(v_v)G_\lambda(u_\lambda), \tag{C.5}$$

where

$$G_\lambda\left(u_\lambda + \frac{r}{2}\right) = \frac{1}{x^{-1} - x} \prod_{j=0}^{\mu-1} f(u_{j+1} - u_j, a_{j\mu}) \prod_{\substack{j=0 \\ j \neq \mu}}^{n-1} [a_{j\mu}]^{-1} \\ \times \prod_{j'=0}^{v-1} f^*(v_{j'+1} - v_{j'}, \xi_{j'v}) \Big|_{v_\lambda = u_\lambda + \frac{1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}}.$$

In the present case, there are at most four poles at  $u_\lambda = u_{\lambda\pm 1} \pm \frac{r-1}{2}$ ,  $v_{\lambda\pm 1} \pm \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}$ , inside the contour for  $z'_\lambda$ -integration. The residues at those four points vanish because of (B.42) and (B.40), respectively. (When  $\lambda = 1$ , we also use (B.43) and (B.41) as well as (B.42) and (B.40).) Therefore, we prove (3.14) for  $\mu \geq 2$ .

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